§6 诺特定理

一、概述

场系统的诺特定理表述为：系统的每一个连续对称性都意味着存在一个守恒流 $\mathcal{J}^\mu(x)$，从而可导出一个守恒荷 $Q(t)$。

$$\partial_\mu \mathcal{J}^\mu(x) = 0 \implies \frac{dQ(t)}{dt} = 0, \quad \text{where } Q(t) \triangleq \frac{1}{c}\int d^3x\, \mathcal{J}^0(x)$$

- **对称性**：拉格朗日密度在场变换下的不变性或规范不变性。
  - 场的变换可以是场空间内部的变化，例如
    $$A^\alpha(x) \quad \longrightarrow \quad A'^\alpha(x) = A^\alpha(x) + \partial^\alpha \psi(x)$$
  - 场的变换也可以是由时空坐标改变所诱导出的，例如
    $$x^\alpha \quad \longrightarrow \quad x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$$
    可诱导出矢量场 $A^\alpha(x)$ 的如下变换
    $$A^\alpha(x) \quad \longrightarrow \quad A'^\alpha(x') = \Lambda^\alpha{}_\beta A^\beta(x)$$

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- 依赖于某些参数的场变换，如果这些参数可以连续取值，并且当这些参数取合适的数值时场是不变的（譬如这些参数均取零，为恒等变换），则这样的变换称为**连续变换**。
  - 譬如，以 ϵ 为参数的如下变换
    $$A^\alpha(x) \quad \longrightarrow \quad A'^\alpha(x) = A^\alpha(x) + \epsilon \partial^\alpha \psi(x)$$
    通常将无穷小的变换记为
    $$A^\alpha(x) \quad \longrightarrow \quad A'^\alpha(x) = A^\alpha(x) + \delta A^\alpha(x)$$
  - 又如，正规洛伦兹变换（6个参数）或时空平移（4个参数）诱导的场变换。
- 系统的**连续对称性**指的就是拉格朗日密度在场的连续变换下的不变性或规范不变性。
  - 连续变换也就是存在无穷小形式的变换。

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二、内部对称性与守恒定律

- 考察张量场 $\varphi_I(x)$ 的无穷小改变

$$\varphi_I(x) \quad \longrightarrow \quad \varphi_I'(x) = \varphi_I(x) + \delta\varphi_I(x)$$

  - 由此引起的拉格朗日密度的改变为

$$\delta\mathcal{L} \triangleq \mathcal{L}(\varphi'(x), \partial\varphi'(x), x) - \mathcal{L}(\varphi(x), \partial\varphi(x), x)$$

- 若在该变换下，$\mathcal{L}$ 在规范含义下不变（规范不变的），即

$$\delta\mathcal{L} = \partial_\mu C^\mu$$

则对于满足拉格朗日方程的场 $\varphi_I(x)$，如下4-矢量场是守恒流

$$\mathcal{J}^\mu(x) \triangleq \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)}\delta\varphi_I - C^\mu, \quad \partial_\mu\mathcal{J}^\mu(x) = 0$$

  - 这样的变换称为 $\mathcal{L}$ 的对称变换。
  - $\mathcal{L}$ 不变（$\delta\mathcal{L} = 0$）是规范不变的特例（$C^\mu = 0$）。

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证明

- 由 $\delta\mathcal{L}$ 的定义：$\delta\mathcal{L} \triangleq \mathcal{L}(\varphi'(x), \partial\varphi'(x), x) - \mathcal{L}(\varphi(x), \partial\varphi(x), x)$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_I}\delta\varphi_I + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)}\boxed{\delta(\partial_\mu\varphi_I)} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)}\delta\varphi_I\right] + \boxed{\frac{\delta\mathcal{L}}{\delta\varphi_I}}\delta\varphi_I$$

- 若相应的变换是 $\mathcal{L}$ 的对称变换，即 $\delta\mathcal{L} = \partial_\mu C^\mu$，则

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)}\delta\varphi_I\right] + \frac{\delta\mathcal{L}}{\delta\varphi_I}\delta\varphi_I = \partial_\mu C^\mu$$

- 对于满足拉格朗日方程的场 φ_I，有

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)}\delta\varphi_I - C^\mu\right] = 0$$

$$\begin{cases} \varphi_I(x) \longrightarrow \varphi_I'(x) = \varphi_I(x) + \delta\varphi_I(x) \\ \partial_\mu\varphi_I(x) \longrightarrow \partial_\mu\varphi_I'(x) = \partial_\mu\varphi_I(x) + \boxed{\partial_\mu\delta\varphi_I(x)} \end{cases}$$

$\delta(\partial_\mu\varphi_I)$，

$$\frac{\delta\mathcal{L}}{\delta\varphi_I} \triangleq \frac{\partial\mathcal{L}}{\partial\varphi_I} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)}$$

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【例】克莱因-戈登场：时空中的复标量场 $\Phi = \varphi_1 + i\varphi_2$，在无电磁耦合情形下，其拉格朗日密度为

$$\mathcal{L}_{\mathrm{KG}} = -\frac{1}{2}\left[\partial_\mu\Phi^*\partial^\mu\Phi + \left(\frac{mc}{\hbar}\right)^2|\Phi|^2\right]$$

$$\Rightarrow \quad \delta\mathcal{L}_{\mathrm{KG}} = -\left[\partial^\mu\Phi\boxed{\delta(\partial_\mu\Phi^*)} + \partial^\mu\Phi^*\delta(\partial_\mu\Phi) + \left(\frac{mc}{\hbar}\right)^2(\Phi\boxed{\delta\Phi^*} + \Phi^*\delta\Phi)\right]$$

- 运动方程

$$\underbrace{\boxed{\frac{\partial\mathcal{L}}{\partial\Phi^*}}}_{-(mc/\hbar)^2\Phi} - \partial_\mu\underbrace{\boxed{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}}}_{-\partial^\mu\Phi} = 0 \quad \Rightarrow \quad \partial_\mu\partial^\mu\Phi - \left(\frac{mc}{\hbar}\right)^2\Phi = 0$$

$$\mathcal{L}_{\mathrm{KG}} = -\frac{1}{2}\left[\partial_\mu\varphi_1\partial^\mu\varphi_1 + \partial_\mu\varphi_2\partial^\mu\varphi_2 + \left(\frac{mc}{\hbar}\right)^2(\varphi_1^2 + \varphi_2^2)\right]$$

$$\Rightarrow \quad \delta\mathcal{L}_{\mathrm{KG}} = -\left[\partial^\mu\varphi_1\delta(\partial_\mu\varphi_1) + \partial^\mu\varphi_2\delta(\partial_\mu\varphi_2) + \left(\frac{mc}{\hbar}\right)^2(\varphi_1\delta\varphi_1 + \varphi_2\delta\varphi_2)\right]$$

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- $\mathcal{L}_{\mathrm{KG}}$ 在如下**整体规范变换**下严格不变（其中 $\alpha \in \mathbb{R}$ ）

$$\begin{cases}\Phi(x) \longmapsto e^{i\alpha}\Phi(x) \\ \Phi^*(x) \longmapsto e^{-i\alpha}\Phi^*(x)\end{cases} \Leftrightarrow \begin{cases}\varphi_1 \longmapsto \varphi_1 \cos\alpha - \varphi_2 \sin\alpha \\ \varphi_2 \longmapsto \varphi_1 \sin\alpha + \varphi_2 \cos\alpha\end{cases}$$

$$\Rightarrow \quad \delta\Phi(x) = i\alpha\Phi(x), \quad \delta\Phi^*(x) = -i\alpha\Phi^*(x)$$

$$\Rightarrow \quad \mathcal{J}^\mu(x) = \boxed{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}}\delta\Phi + \boxed{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}}\delta\Phi^* \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} = -\frac{1}{2}\partial^\mu\Phi^* \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)} = -\frac{1}{2}\partial^\mu\Phi$$

$$= \frac{1}{2}i\alpha(\Phi^*\partial^\mu\Phi - \Phi\partial^\mu\Phi^*)$$

乘上合适倍数对守恒流重新定义，可使其具有电流密度量纲

$$J^\mu(x) = iec[\Phi^*(x)\partial^\mu\Phi(x) - \Phi(x)\partial^\mu\Phi^*(x)] = -ec\,\mathrm{Im}[\Phi^*(x)\partial^\mu\Phi(x)]$$

$$\mathcal{L}_{\mathrm{KG}} = -\frac{1}{2}[\partial_\mu\Phi^*\partial^\mu\Phi + (mc/\hbar)^2|\Phi|^2]$$

$$\mathcal{L}_{\mathrm{KG}} = -\frac{1}{2}[\partial_\mu\varphi_1\partial^\mu\varphi_1 + \partial_\mu\varphi_2\partial^\mu\varphi_2 + (mc/\hbar)^2(\varphi_1^2 + \varphi_2^2)]$$

【例】规范变换与电荷守恒

设 $A_\mu(x)$ 为电磁势，考察规范变换

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\psi(x)$$

在规范变换下，电磁场的拉格朗日密度变换为

$$\mathcal{L}(A', \partial A', x) = \mathcal{L}(A, \partial A, x) + \boxed{J^\mu(x)\partial_\mu\psi(x)}$$

$$\partial_\mu(\psi J^\mu) - \psi\partial_\mu J^\mu$$

由于 $J^\mu(x)$ 是守恒流，即 $\partial_\mu J^\mu(x) = 0$，因而

$$\delta\mathcal{L} = J^\mu(x)\partial_\mu\psi(x) = \partial_\mu[\psi(x)J^\mu(x)]$$

当且仅当电流密度 $J^\mu(x)$ 满足 $\partial_\mu J^\mu(x) = 0$ 时，

电磁场与带电物质的耦合项 $J^\mu(x)A_\mu(x)$ 是规范不变的，

$$\mathcal{L}(A, \partial A, x) = -\frac{1}{4\mu_0}F_{\alpha\beta}(x)F^{\alpha\beta}(x) + J^\mu(x)A_\mu(x)$$

三、无穷小的庞加莱变换

- **无穷小庞加莱变换：** $x'^\alpha = \boxed{x^\alpha + \Omega^\alpha{}_\beta x^\beta} + a^\alpha$ $\qquad \left(\delta^\alpha_\beta + \Omega^\alpha{}_\beta\right)x^\beta = \Lambda^\alpha{}_\beta x^\beta$

  其中，$\Omega^\alpha{}_\beta$ 和 a^α 均为无穷小参数，且

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}, \quad \text{where } \omega_{\alpha\beta} \triangleq g_{\alpha\mu}\Omega^\mu{}_\beta$$

- 定义**坐标的变分：** $\delta x^\alpha \triangleq x'^\alpha - x^\alpha = \Omega^\alpha{}_\beta x^\beta + a^\alpha$

$$\Rightarrow \quad \partial_\alpha\delta x^\alpha = \Omega^\alpha{}_\beta\boxed{\partial_\alpha x^\beta} = \Omega^\alpha{}_\alpha \qquad (\partial_\alpha x^\beta = \delta^\alpha_\beta)$$

$$\Omega = \begin{pmatrix} 0 & -\vec{\xi} \\ -\vec{\xi} & \overleftrightarrow{\theta} \end{pmatrix}$$

$$\Rightarrow \quad \partial_\alpha\delta x^\alpha = 0$$

- 根据张量场的定义：

$$T'^{\alpha\cdots\beta}(x') = \Lambda^\alpha{}_\mu\cdots\Lambda^\beta{}_\nu T^{\mu\cdots\nu}(x)$$

$$\Rightarrow \quad T'^{\alpha\cdots\beta}(x') = \left(\delta^\alpha_\mu + \Omega^\alpha{}_\mu\right)\cdots\left(\delta^\beta_\nu + \Omega^\beta{}_\nu\right)T^{\mu\cdots\nu}(x)$$

1. 场的变分

- 张量场 φ_I 的**全变分**定义为（关于 δx 的一阶改变）：

$$\bar{\delta}\varphi_I(x) \triangleq \varphi_I'(x') - \varphi_I(x)$$

  - $\bar{\delta}\varphi_I(x)$： 即包含张量变换引起的场本身的改变，也包含时空坐标变换引起的场的改变。
  - 主动观点： $\bar{\delta}\varphi_I(x)$ 是两个不同张量场 φ_I' 和 φ_I 在两个不同时空点 x' 和 x 的数值之差。

- 张量场 φ_I 的**形式变分**定义为（关于 δx 的一阶改变）：

$$\delta\varphi_I(x) \triangleq \varphi_I'(x) - \varphi_I(x)$$

  - $\delta\varphi_I(x)$ 反映同一时空坐标处张量变换引起的场的改变。
  - 主动观点： $\delta\varphi_I(x)$ 是两个不同张量场 φ_I' 和 φ_I 在同一时空点 x 的数值之差。

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场的全变分： $\bar{\delta}\varphi_I(x) \triangleq \varphi_I'(x') - \varphi_I(x)$

- 标量场的全变分：$\bar{\delta}\varphi(x) = 0$

$$\bar{\delta}\varphi(x) = \varphi'(x') - \varphi(x)$$

- 矢量场的全变分：$\bar{\delta}A^\alpha(x) = \Omega^\alpha{}_\mu A^\mu(x)$

$$\bar{\delta}A^\alpha(x) = \boxed{A'^\alpha(x')} - A^\alpha(x)$$

$$\left(\delta^\alpha_\mu + \Omega^\alpha{}_\mu\right)A^\mu(x)$$

- 张量场的全变分：$\bar{\delta}T^{\alpha\beta}(x) = \Omega^\alpha{}_\mu T^{\mu\beta}(x) + \Omega^\beta{}_\nu T^{\alpha\nu}(x)$

$$\bar{\delta}T^{\alpha\beta}(x) = \boxed{T'^{\alpha\beta}(x')} - T^{\alpha\beta}(x)$$

$$\left(\delta^\alpha_\mu + \Omega^\alpha{}_\mu\right)\left(\delta^\beta_\nu + \Omega^\beta{}_\nu\right)T^{\mu\nu}(x)$$

$$T'^{\alpha\cdots\beta}(x') = \left(\delta^\alpha_\mu + \Omega^\alpha{}_\mu\right)\cdots\left(\delta^\beta_\nu + \Omega^\beta{}_\nu\right)T^{\mu\cdots\nu}(x)$$

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场的全变分与形式变分的关系

- 根据形式变分与全变分定义：

$$\delta\varphi_I(x) = -\boxed{[\varphi_I'(x') - \varphi_I'(x)]} + \boxed{[\varphi_I'(x') - \varphi_I(x)]}$$

$$\delta x_\mu \partial^\mu \varphi_I(x) \qquad\qquad \bar{\delta}\varphi_I(x)$$

$$\Longrightarrow \quad \delta\varphi_I(x) = \bar{\delta}\varphi_I(x) - \delta x_\mu \partial^\mu \varphi_I(x)$$

- 根据形式变分的定义

$$\delta\frac{\partial\varphi_I(x)}{\partial x^\alpha} = \frac{\partial\varphi_I'(x)}{\partial x^\alpha} - \frac{\partial\varphi_I(x)}{\partial x^\alpha} \quad \Longrightarrow \quad \delta(\partial_\alpha\varphi_I) = \partial_\alpha(\delta\varphi_I)$$

- 根据全变分的定义

$$\bar{\delta}\frac{\partial\varphi_I(x)}{\partial x^\alpha} = \frac{\partial\varphi_I'(x')}{\partial x'^\alpha} - \frac{\partial\varphi_I(x)}{\partial x^\alpha} \quad \Longrightarrow \quad \bar{\delta}(\partial_\alpha\varphi_I) \neq \partial_\alpha(\bar{\delta}\varphi_I)$$

$$\bar{\delta}\varphi_I(x) \triangleq \varphi_I'(x') - \varphi_I(x), \qquad \delta\varphi_I(x) \triangleq \varphi_I'(x) - \varphi_I(x)$$

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2. 拉格朗日密度的变分

- 拉格朗日密度 $\mathcal{L} = \mathcal{L}(\varphi, \partial\varphi, x)$ 的**变分**定义为：

$$\delta\mathcal{L}(\varphi, \partial\varphi, x) \triangleq \mathcal{L}(\varphi'(x'), \partial'\varphi'(x'), x') - \mathcal{L}(\varphi(x), \partial\varphi(x), x)$$

- 若在变换 $x^\alpha \longrightarrow x'^\alpha = x^\alpha + \delta x^\alpha$ 下，$\mathcal{L}$ 是**规范不变**的，即

$$\delta\mathcal{L} = \partial_\mu C^\mu$$

则称该变换是 $\mathcal{L}$ 的**对称变换**。

➢ $\mathcal{L}$ 不变（$\delta\mathcal{L} = 0$）是规范不变的特例（$C^\mu = 0$）。

➢ $\mathcal{L}$ 的每一个连续对称性都意味着体系存在一个守恒流 $\mathcal{J}^\nu$：即对于满足拉格朗日方程的场有

$$\partial_\nu \mathcal{J}^\nu = 0$$

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拉格朗日密度的全变分

- 根据拉格朗日密度变分的定义

$$\delta x^\nu \partial_\nu \mathcal{L}(\varphi'(x), \partial\varphi'(x), x) = \delta x^\nu \partial_\nu \mathcal{L}$$

$$\delta\mathcal{L} = \boxed{[\mathcal{L}(\varphi'(x'), \partial'\varphi'(x'), x') - \mathcal{L}(\varphi'(x), \partial\varphi'(x), x)]}$$

$$+[\mathcal{L}(\boxed{\varphi'(x)}, \boxed{\partial\varphi'(x)}, x) - \mathcal{L}(\varphi(x), \partial\varphi(x), x)]$$

$$\varphi(x) + \delta\varphi(x) \qquad \partial\varphi(x) + \partial\delta\varphi(x)$$

$$\Longrightarrow \quad \delta\mathcal{L} = \delta x^\nu \partial_\nu \mathcal{L} + \left[\frac{\partial\mathcal{L}}{\partial\varphi_I}\delta\varphi_I + \boxed{\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi_I)}\partial_\nu(\delta\varphi_I)}\right]$$

$$\partial_\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi_I)}\delta\varphi_I\right] - \left[\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi_I)}\right]\delta\varphi_I$$

➢ $\partial_\nu\mathcal{L}$ 是 $\mathcal{L}$ 对 x^ν 的全偏导数：固定其他时空坐标对 x^ν 求偏导数。而非固定与场相关的量 $(\varphi, \partial\varphi)$，对 x^ν 求偏导数。

$$\delta\mathcal{L}(\varphi, \partial\varphi, x) \triangleq \mathcal{L}(\varphi'(x'), \partial'\varphi'(x'), x') - \mathcal{L}(\varphi(x), \partial\varphi(x), x)$$

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拉格朗日密度在真实场处的全变分

- 对满足拉格朗日方程的场 $\varphi_I(x)$，$\mathcal{L} = \mathcal{L}(\varphi, \partial\varphi, x)$ 的变分为

$$\delta\mathcal{L} = \boxed{\delta x^\nu \partial_\nu \mathcal{L}} + \partial_\nu\left(\Pi^{\nu I}\delta\varphi_I\right), \qquad \Pi^{\nu I} \triangleq \frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi_I)}$$

$$\partial_\nu(\mathcal{L}\delta x^\nu)$$

$$\bar{\delta}\varphi_I - \delta x_\mu \partial^\mu \varphi_I$$

$$\Longrightarrow \quad \delta\mathcal{L} = \partial_\nu\left(\mathcal{L}\delta x^\nu + \Pi^{\nu I}\boxed{\delta\varphi_I}\right)$$

$$= \partial_\nu\left[\mathcal{L}\boxed{\delta x^\nu} - \Pi^{\nu I}(\partial^\mu\varphi_I)\delta x_\mu + \Pi^{\nu I}\bar{\delta}\varphi_I\right]$$

$$g^{\mu\nu}\delta x_\mu$$

$$T^{\mu\nu}$$

$$= \partial_\nu \boxed{\left\{\boxed{\left[g^{\mu\nu}\mathcal{L} - \Pi^{\nu I}\partial^\mu\varphi_I\right]}\delta x_\mu + \Pi^{\nu I}\bar{\delta}\varphi_I\right\}}$$

$$\mathcal{J}^\nu$$

$$\Longrightarrow \quad \delta\mathcal{L} = \partial_\nu \mathcal{J}^\nu$$

$$\delta\mathcal{L} = \delta x^\nu \partial_\nu \mathcal{L} + \partial_\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi_I)}\delta\varphi_I\right] + \left[\frac{\partial\mathcal{L}}{\partial\varphi_I} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi_I)}\right]\delta\varphi_I, \quad \partial_\alpha\delta x^\alpha = 0$$

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诺特定理

若系统的拉格朗日密度 $\mathcal{L}$ 在连续庞加莱变换下不变，即

$$\delta\mathcal{L}(\varphi,\partial\varphi,x) \triangleq \mathcal{L}(\varphi'(x'),\partial'\varphi'(x'),x') - \mathcal{L}(\varphi(x),\partial\varphi(x),x) = 0$$

则 $\mathcal{J}^\nu$ 为守恒流（$\partial_\nu\mathcal{J}^\nu = 0$），其中

$$\begin{cases} \mathcal{J}^\nu \triangleq T^{\mu\nu}\delta x_\mu + \Pi^{\nu I}\bar{\delta}\varphi_I \\ T^{\mu\nu} \triangleq g^{\mu\nu}\mathcal{L} - \Pi^{\nu I}\partial^\mu\varphi_I \end{cases}$$

- 连续庞加莱变换：正规洛伦兹变换及时空平移变换。
- 标量场、矢量场的全变分分别为

$$\bar{\delta}\varphi = 0, \quad \bar{\delta}A^\alpha = \Omega^\alpha{}_\mu A^\mu \quad \Longrightarrow \quad \bar{\delta}A_\alpha = \omega_{\alpha\mu}A^\mu$$

$$\delta x_\alpha = \omega_{\alpha\mu}x^\mu + a_\alpha, \quad \bar{\delta}\varphi_I(x) \triangleq \varphi_I'(x') - \varphi_I(x)$$

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四、时空平移不变性

- 由于在时空平移变换 $x_\alpha \longrightarrow x_\alpha' = x_\alpha + a_\alpha$ 下

$$\mathcal{L}(\varphi'(x'),\partial'\varphi'(x'),x') = \mathcal{L}(\varphi(x),\partial\varphi(x),x+a)$$

因此，$\mathcal{L}$ 在平移变换下是不变的（$\delta\mathcal{L} = 0$）等价于说

$$\mathcal{L}(\varphi(x),\partial\varphi(x),x+a) = \mathcal{L}(\varphi(x),\partial\varphi(x),x)$$

仅当 $\mathcal{L}$ 不明显依赖于时空坐标 x 时，它才是平移不变的。

- 在时空平移变换下

$$\delta x_\mu = a_\mu, \quad \bar{\delta}\varphi_I = 0 \quad \Longrightarrow \quad \mathcal{J}^\nu = T^{\mu\nu}\delta x_\mu = T^{\mu\nu}a_\mu$$

$$\Longrightarrow \quad \delta\mathcal{L} = \partial_\nu\mathcal{J}^\nu = a_\mu\partial_\nu T^{\mu\nu}$$

$$\delta\mathcal{L} = 0, \quad (\forall a_\mu) \quad \Longrightarrow \quad \partial_\nu T^{\mu\nu} = 0$$

$$\delta\mathcal{L} = \partial_\nu\mathcal{J}^\nu, \quad \mathcal{J}^\nu \triangleq T^{\mu\nu}\delta x_\mu + \Pi^{\nu I}\bar{\delta}\varphi_I, \quad T^{\mu\nu} \triangleq g^{\mu\nu}\mathcal{L} - \Pi^{\nu I}\partial^\mu\varphi_I$$

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1. 时空平移不变性与四维动量守恒

若时空平移变换下拉格朗日密度是不变的、
亦即若拉格朗日密度不显含时空坐标，则

$$\partial_\nu T^{\mu\nu} = 0$$

- $T^{\mu\nu}$ 称为场的**正则能量-动量张量**

$$T^{\mu\nu} \triangleq g^{\mu\nu}\mathcal{L} - (\partial^\mu\varphi_I)\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi_I)}$$

- 与 $T^{\mu\nu}$ 对应的四个守恒荷称为系统的**总四维动量**

$$P^\mu \triangleq \frac{1}{c}\int T^{\mu 0}d^3x = \left(\frac{\mathcal{E}}{c}, \ \vec{P}\right)$$

- **思考：** $T^{\mu\nu}$ 为守恒流也可利用场方程加以证明。

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分量 T^{00} 的含义

$$T^{00}=g^{00}\mathcal{L}-\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi_I)}\partial^0\varphi_I$$
$$=\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi_I)}\partial_t\varphi_I-\mathcal{L}$$
$$\Rightarrow\quad \mathcal{E}=\int T^{00}d^3x=\int\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi_I)}\partial_t\varphi_I d^3x-L$$

类比离散体系的哈密顿函数

$$H=\frac{\partial L}{\partial\dot{q}_k}\dot{q}_k-L$$

可见，$\mathcal{E}$ 可诠释为是场的总能量。

$$T^{\mu\nu}\triangleq g^{\mu\nu}\mathcal{L}-(\partial^\mu\varphi_I)\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi_I)}$$

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标量场、矢量场的正则能量-动量张量

- 对于标量场，正则能量-动量张量表示为

$$T^{\mu\nu}=g^{\mu\nu}\mathcal{L}-\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi)}\partial^\mu\varphi$$

【思考】自由标量场的 $\mathcal{L}$ 如下，请写出 $T^{\mu\nu}$ 以及 T^{00}

$$\mathcal{L}=-\frac{1}{2}\left[\partial_\mu\varphi\partial^\mu\varphi+\left(\frac{mc}{\hbar}\right)^2\varphi^2\right]$$

- 对于矢量场，正则能量-动量张量表示为

$$T^{\mu\nu}=g^{\mu\nu}\mathcal{L}-\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\alpha)}\partial^\mu A_\alpha$$

【思考】请写出二阶张量场的正则能量-动量张量。

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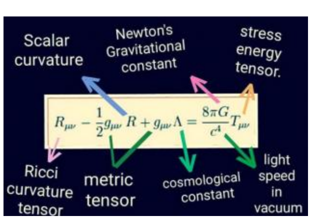
能量-动量张量的对称性和不确定性

- 正则能量-动量张量 $T^{\mu\nu}$ 可能是不对称的。
  - **对称能量-动量密度张量的存在是任何物理系统与引力耦合的不可或缺的先决条件。**



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- 总可通过加上一个附加项 $t^{\mu\nu}$，使得 $T^{\mu\nu}+t^{\mu\nu}$ 为对称张量。
  - **附加项不应导致守恒定律发生改变。**即 $t^{\mu\nu}$ 须满足

$$\partial_\nu t^{\mu\nu}=0$$

  - **附加项不应导致总的能量和动量发生改变。**即 $t^{\mu\nu}$ 须满足

$$\int d^3x\, t^{\mu 0}=0$$

  - **附加项使得四维动量重新分布，但却不改变其总值。**
- $T^{\mu\nu}$ 中可添加附加项表明：能量-动量张量的定义具有模糊性。
  - 这一模糊性只能依靠引进某些附加的假设才能被消除。
  - 在电磁理论中可假设：能量-动量张量应该是规范不变的。

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2. 电磁场的能量-动量张量

- 自由电磁场的拉格朗日密度具有时空平移不变性

$$\mathcal{L}_f=-\frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta}=\mathcal{L}_f(\partial A)$$

- 电磁场的正则能量-动量张量 $T^{\mu\nu}$ 为

$$T^{\mu\nu}=g^{\mu\nu}\mathcal{L}_f-\boxed{\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\alpha)}}\partial^\mu A_\alpha$$

$$\Longrightarrow\quad T^{\mu\nu}=g^{\mu\nu}\mathcal{L}_f+\frac{1}{\mu_0}F^{\nu\alpha}\partial^\mu A_\alpha$$

  - $T^{\mu\nu}$ 满足连续性方程：$\partial_\nu T^{\mu\nu}=0$。
  - $T^{\mu\nu}$ 并非对称的。这一缺陷可通过添加附加项弥补。

$$T^{\mu\nu}\triangleq g^{\mu\nu}\mathcal{L}-(\partial^\mu\varphi_l)\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi_l)},\qquad \frac{\partial\mathcal{L}_f}{\partial(\partial_\nu A_\alpha)}=-\frac{1}{\mu_0}F^{\nu\alpha}$$

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附加项

- 将电磁场的正则能量-动量张量改写为

$$T^{\mu\nu}=g^{\mu\nu}\mathcal{L}+\frac{1}{\mu_0}F^{\nu\alpha}\underbrace{\boxed{(\partial^\mu A_\alpha-\partial_\alpha A^\mu)}}_{F^\mu{}_\alpha}+\frac{1}{\mu_0}\underbrace{\boxed{F^{\nu\alpha}\partial_\alpha A^\mu}}_{\partial_\alpha(A^\mu F^{\nu\alpha})-A^\mu(\partial_\alpha F^{\nu\alpha})}$$

利用自由电磁场的麦克斯韦方程 $\partial_\mu F^{\mu\alpha}=0$，得

$$T^{\mu\nu}=g^{\mu\nu}\mathcal{L}-\frac{1}{\mu_0}F^{\mu\alpha}F_\alpha{}^\nu+\underbrace{\boxed{\partial_\alpha\left(\frac{1}{\mu_0}A^\mu F^{\nu\alpha}\right)}}_{t^{\mu\nu}}$$

- $t^{\mu\nu}$ 项不影响守恒定律，也不影响守恒荷的数值。

$$\begin{cases}\partial_\nu t^{\mu\nu}=\dfrac{1}{\mu_0}\partial_\nu\partial_\alpha(A^\mu F^{\nu\alpha})=0\\[2mm] \displaystyle\int d^3x\, t^{\mu0}=\frac{1}{\mu_0}\int d^3x\underbrace{\boxed{\partial_\alpha\left(A^\mu F^{0\alpha}\right)}}_{\partial_k(A^\mu F^{0k})}=\frac{1}{\mu_0}\oint d\sigma_k\left(A^\mu F^{0k}\right)=0\end{cases}$$

$$T^{\mu\nu}=g^{\mu\nu}\mathcal{L}+\frac{1}{\mu_0}F^{\nu\alpha}\partial^\mu A_\alpha$$

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电磁场的能量-动量张量

- 将附加项去掉，定义 $T_f^{\mu\nu} \triangleq T^{\mu\nu} - t^{\mu\nu}$，亦即

$$T_f^{\mu\nu} = g^{\mu\nu}\mathcal{L}_f - \frac{1}{\mu_0}F^{\mu}{}_{\alpha}F^{\alpha\nu} = \begin{pmatrix} w & c\vec{g} \\ c\vec{g} & \overleftrightarrow{T} \end{pmatrix}$$

这就是**电磁场的能量-动量张量**。

- ➢ $T_f^{\mu\nu}$ 和 $T^{\mu\nu}$ 的守恒荷相同，均给出电磁场的四维动量：

$$P^{\mu} \triangleq \frac{1}{c}\int T_f^{\mu 0} d^3x = \left(\frac{W}{c},\ \ \vec{G}\right) \triangleq \left(\frac{\mathcal{E}_f}{c},\ \ \vec{P}_f\right)$$

- ➢ **$T_f^{\mu\nu}$：对称、无迹、规范不变。**

$$\mathcal{L}_f = -\frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta}$$

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能量-动量张量在数学上的不确定性

- 能量动量张量在满足下面的条件下是否还具有不确定性？

$$\partial_\nu T_f^{\mu\nu} = 0, \quad T_f^{\mu\nu} = T_f^{\nu\mu}$$

- 数学上，仍可以添加合适的附加项

$$\tilde{T}_f^{\mu\nu} = T_f^{\mu\nu} + \underbrace{\partial_\alpha \Omega^{\mu\nu\alpha}}_{t^{\mu\nu}}$$

- ➢ 附加项应使得

只需 $\Omega^{\mu\nu\alpha} = -\Omega^{\mu\alpha\nu}$

$$\begin{cases} \partial_\nu t^{\mu\nu} = 0 \\ t^{\mu\nu} = t^{\nu\mu} \end{cases} \Rightarrow \begin{cases} \partial_\nu \partial_\alpha \Omega^{\mu\nu\alpha} = 0 \\ \Omega^{\mu\nu\alpha} = \Omega^{\nu\mu\alpha} \end{cases}$$

- ➢ 这样的附加项自动不改变守恒荷的数值

$$\int d^3x\, t^{\mu 0} \propto \int d^3x \underbrace{\partial_\alpha \Omega^{\mu 0 \alpha}}_{\partial_k \Omega^{\mu 0 k}} = \oint d\sigma_k\, \Omega^{\mu 0 k} = 0$$

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能量-动量张量在物理上的唯一性

- 在满足对称、守恒、规范不变前提下，$T_f^{\mu\nu}$ 是否唯一？
  - ➢ 即 $T_f^{\mu\nu}$ 是否还可以添加附加项 $t^{\mu\nu}$，这里二阶张量场 $t^{\mu\nu}$：规范不变、且不改变守恒荷的数值、还满足

$$\partial_\nu t^{\mu\nu} = 0, \quad t^{\mu\nu} = t^{\nu\mu}$$

- 规范不变性要求附加的二阶张量场 $t^{\mu\nu}$ 只能由如下张量构造

$$F^{\alpha\beta}, \quad g^{\alpha\beta}, \quad g_{\alpha\beta}$$

- ➢ 量纲匹配意味着 $t^{\mu\nu}$ 应是 $F^{\alpha\beta}$ 的二次函数。
- ➢ 符合要求的 $t^{\mu\nu}$ 只可能具有如下形式

$$t^{\mu\nu} = a_1 \underbrace{\left(-\frac{1}{4\mu_0}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)}_{\text{对称}} + a_2 \underbrace{\left(-\frac{1}{\mu_0}F^{\mu\alpha}F_{\alpha}{}^{\nu}\right)}_{\text{对称}} + a_3 F^{\mu\nu}F_{\alpha}{}^{\alpha}$$

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- 符合对称、规范不变性要求的附加项只可能具有下面的形式

$$t^{\mu\nu} = b_1 T_f^{\mu\nu} + b_2\left(-\frac{1}{4\mu_0} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{\mu_0} F^{\mu\alpha}F_\alpha{}^\nu\right)$$

- $t^{\mu\nu}$ 不影响守恒定律意味着

$4(\partial^\nu F^{\mu\alpha})F_{\alpha\nu} = -2F_{\alpha\beta}\partial^\mu F^{\alpha\beta}$

$$0 = \partial_\nu t^{\mu\nu} = \frac{1}{4\mu_0} b_2\left[-2F_{\alpha\beta}\partial^\mu F^{\alpha\beta} + 4(\partial_\nu F^{\mu\alpha})F_\alpha{}^\nu + 4F^{\mu\alpha}\partial_\nu F_\alpha{}^\nu\right]$$

$$\Rightarrow \quad 0 = -\frac{1}{\mu_0} b_2 F_{\alpha\beta}\partial^\mu F^{\alpha\beta} \quad \Rightarrow \quad b_2 = 0 \quad \Rightarrow \quad t^{\mu\nu} = b_1 T_f^{\mu\nu}$$

- $t^{\mu\nu}$ 不能改变守恒荷意味着 $b_1 = 0$。

在要求规范不变性的前提下，$T_f^{\mu\nu}$ 的定义是唯一的。

$$F_{\alpha\nu}\partial^\nu F^{\mu\alpha} = -F_{\alpha\nu}(\partial^\mu F^{\alpha\nu} + \partial^\alpha F^{\nu\mu}) = -F_{\alpha\beta}\partial^\mu F^{\alpha\beta} - F_{\nu\alpha}\partial^\nu F^{\alpha\mu} \quad F_{\alpha\nu}\partial^\nu F^{\mu\alpha}$$

$$\Rightarrow \quad 2F_{\alpha\nu}\partial^\nu F^{\mu\alpha} = -F_{\alpha\beta}\partial^\mu F^{\alpha\beta}$$

五、洛伦兹变换下的不变性

下面只针对矢量场 A^α 讨论。

$$\begin{cases} T^{\mu\nu} = g^{\mu\nu}\mathcal{L} - \Pi^{\nu\alpha}\partial^\mu A_\alpha \\ \mathcal{J}^\nu = T^{\mu\nu}\delta x_\mu + \Pi^{\nu\alpha}\bar{\delta}A_\alpha = \omega_{\alpha\beta}(x^\beta T^{\alpha\nu} + A^\beta \Pi^{\nu\alpha}) \end{cases}$$

$\delta x_\mu = \omega_{\mu\beta}x^\beta$, $\bar{\delta}A_\alpha = \omega_{\alpha\beta}A^\beta$, $= -\omega_{\alpha\beta}(x^\alpha T^{\beta\nu} + A^\alpha \Pi^{\nu\beta})$

$$\Rightarrow \quad \mathcal{J}^\nu = \frac{1}{2}\omega_{\alpha\beta}\left[(x^\beta T^{\alpha\nu} - x^\alpha T^{\beta\nu}) + (A^\beta \Pi^{\nu\alpha} - A^\alpha \Pi^{\nu\beta})\right]$$

$$\Rightarrow \quad \mathcal{J}^\nu = -\frac{1}{2}\omega_{\alpha\beta}\left[(x^\alpha T^{\beta\nu} - x^\beta T^{\alpha\nu}) + (A^\alpha \Pi^{\nu\beta} - A^\beta \Pi^{\nu\alpha})\right]$$

$$M^{\alpha\beta\nu} = (x^\alpha T^{\beta\nu} - A^\beta \Pi^{\nu\alpha}) - (\alpha \leftrightarrow \beta)$$

$$\Rightarrow \quad \delta\mathcal{L} = \partial_\nu \mathcal{J}^\nu = -\frac{1}{2}\omega_{\alpha\beta}\partial_\nu M^{\alpha\beta\nu}$$

$\delta\mathcal{L} = \partial_\nu \mathcal{J}^\nu$, $\quad \mathcal{J}^\nu \triangleq T^{\mu\nu}\delta x_\mu + \Pi^{\nu I}\bar{\delta}\varphi_I$, $\quad T^{\mu\nu} \triangleq g^{\mu\nu}\mathcal{L} - \Pi^{\nu I}\partial^\mu \varphi_I$

洛伦兹不变性与角动量守恒

若洛伦兹变换下拉格朗日密度是不变的，则

$$\partial_\nu M^{\alpha\beta\nu} = 0$$

- $M^{\alpha\beta\nu}$ 称为**正则角动量流密度张量**
  - $M^{\alpha\beta\nu}$ 关于指标 α 和 β 反对称：$M^{\alpha\beta\nu} = -M^{\beta\alpha\nu}$。
  - $M^{\alpha\beta\nu}$ 并不具有标准的形式。
- 与 $M^{\alpha\beta\nu}$ 对应的个守恒荷称为系统的**总4-角动量张量**

$$L^{\alpha\beta} \triangleq \frac{1}{c}\int M^{\alpha\beta 0} d^3x$$

$M^{\alpha\beta\nu} = (x^\alpha T^{\beta\nu} - A^\beta \Pi^{\nu\alpha}) - (\alpha \leftrightarrow \beta)$, $\quad T^{\mu\nu} = g^{\mu\nu}\mathcal{L} - \Pi^{\nu\alpha}\partial^\mu A_\alpha$

电磁场的正则角动量流密度张量

- 自由电磁场的拉格朗日密度在洛伦兹变换下不变。因此

$$\partial_\nu M^{\alpha\beta\nu}=0$$

其中，$M^{\alpha\beta\nu}$ 为电磁场的正则角动量流密度张量。

$$\Pi^{\nu\alpha}=\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\alpha)}=-\frac{1}{\mu_0}F^{\nu\alpha}$$

$$\Rightarrow\quad M^{\alpha\beta\nu}=\left(x^\alpha T^{\beta\nu}+\frac{1}{\mu_0}A^\beta F^{\nu\alpha}\right)-(\alpha\leftrightarrow\beta)$$

$$T^{\beta\nu}=T_f^{\beta\nu}+t^{\beta\nu},\qquad t^{\beta\nu}=\frac{1}{\mu_0}\partial_\mu\left(A^\beta F^{\nu\mu}\right)$$

$$\Rightarrow\quad M^{\alpha\beta\nu}=\left[x^\alpha T_f^{\beta\nu}+\frac{1}{\mu_0}x^\alpha\partial_\mu\left(A^\beta F^{\nu\mu}\right)+\frac{1}{\mu_0}A^\beta F^{\nu\alpha}\right]-(\alpha\leftrightarrow\beta)$$

$M^{\alpha\beta\nu}=\left(x^\alpha T^{\beta\nu}-A^\beta\Pi^{\nu\alpha}\right)-(\alpha\leftrightarrow\beta)$, $T^{\mu\nu}=g^{\mu\nu}\mathcal{L}-\Pi^{\nu\alpha}\partial^\mu A_\alpha$

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- 利用能量-动量张量，可将 $M^{\alpha\beta\nu}$ 表示为

$$M^{\alpha\beta\nu}=\left[x^\alpha T_f^{\beta\nu}+\frac{1}{\mu_0}x^\alpha\partial_\mu\left(A^\beta F^{\nu\mu}\right)+\frac{1}{\mu_0}A^\beta F^{\nu\alpha}\right]-(\alpha\leftrightarrow\beta)$$

$$\Rightarrow\quad M^{\alpha\beta\nu}=x^\alpha T_f^{\beta\nu}+\frac{1}{\mu_0}\partial_\mu\left(x^\alpha A^\beta F^{\nu\mu}\right)-(\alpha\leftrightarrow\beta)$$

$$=\left(x^\alpha T_f^{\beta\nu}-x^\beta T_f^{\alpha\nu}\right)+\underbrace{\frac{1}{\mu_0}\partial_\mu\left[\left(x^\alpha A^\beta-x^\beta A^\alpha\right)F^{\nu\mu}\right]}_{m^{\alpha\beta\nu}}$$

- $m^{\alpha\beta\nu}$ 项不影响守恒定律，也不影响守恒荷的数值。

$$\begin{cases}\partial_\nu m^{\alpha\beta\nu}=\dfrac{1}{\mu_0}\partial_\nu\partial_\mu\left[\left(x^\alpha A^\beta-x^\beta A^\alpha\right)F^{\nu\mu}\right]=0\\[2ex]\displaystyle\int d^3x\,m^{\alpha\beta0}=\frac{1}{\mu_0}\int d^3x\,\partial_\mu\left[\left(x^\alpha A^\beta-x^\beta A^\alpha\right)F^{0\mu}\right]=0\end{cases}$$

$\partial_k\left[\left(x^\alpha A^\beta-x^\beta A^\alpha\right)F^{0k}\right]$

$\partial_\mu\left(A^\beta F^{\nu\mu}\right)=\partial_\mu\left(x^\alpha A^\beta F^{\nu\mu}\right)-\left(\partial_\mu x^\alpha\right)\left(A^\beta F^{\nu\mu}\right)$ $A^\beta F^{\nu\alpha}$

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电磁场的能量-动量张量

- 重新定义**电磁场的角动量流密度张量**:

$$M_f^{\alpha\beta\nu}=x^\alpha T_f^{\beta\nu}-x^\beta T_f^{\alpha\nu}$$

  - $M_f^{\alpha\beta\nu}$ 具有标准形式，且关于 α 和 β 反对称。
  - $M_f^{\alpha\beta\nu}$ 是规范不变的。
  - 守恒荷为

$$L^{\alpha\beta}=\{\vec{K},\ \vec{L}\}=\begin{pmatrix}0 & \vec{K}\\ -\vec{K} & \vec{L}\end{pmatrix}\qquad\begin{cases}\vec{K}=ct\vec{G}-\dfrac{1}{c}\displaystyle\int w\vec{x}d^3x\\[2ex]\vec{L}=\displaystyle\int(\vec{x}\times\vec{g})d^3x\end{cases}$$

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六、电动力学规律的推导

- **相对性原理、规范不变性**决定了场的拉格朗日密度

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_{pf} = -\frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} + J^\alpha A_\alpha$$

其中，电磁场张量的定义为

$$F_{\alpha\beta} \triangleq \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

- ➢ $\mathcal{L}$ 决定了电磁场满足麦克斯韦方程

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$$

- ➢ $F_{\alpha\beta}$ 的定义意味着其满足毕安琪恒等式

$$\partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0$$

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- 由 $\mathcal{L}_f$ 可构造电磁场的正则能量-动量张量

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} + \frac{1}{\mu_0} F^{\nu\alpha} \partial^\mu A_\alpha$$

- ➢ **对称性、规范不变性**唯一确定了电磁场的能量-动量张量

$$T_f^{\mu\nu} = g^{\mu\nu}\mathcal{L} - \frac{1}{\mu_0} F^{\mu\alpha} F_\alpha{}^\nu$$

- 粒子系统的能量-动量张量可直接由其物理意义读出，为

$$T_p^{\alpha\beta} = n_0 m u^\alpha u^\beta = \rho_0 u^\alpha u^\beta$$

- 电磁场、粒子系统的能量-动量张量可分别写为

$$T_f^{\mu\nu} = \begin{pmatrix} w & \vec{S}/c \\ c\vec{g} & \overleftrightarrow{T} \end{pmatrix}, \qquad T_p^{\mu\nu} = \gamma\rho \begin{pmatrix} c^2 & c\vec{v} \\ c\vec{v} & \vec{v}\vec{v} \end{pmatrix}$$

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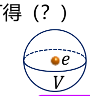
- **带电粒子-电磁场系统的能量-动量守恒**意味着：

$$\partial_\nu T^{\mu\nu} = \partial_\nu \left(T_f^{\mu\nu} + T_p^{\mu\nu} \right) = 0$$

- ➢ 利用麦克斯韦方程和毕安琪恒等式，直接计算可得（？）

$$\partial_\nu T_f^{\mu\nu} = -F^{\mu\alpha} J_\alpha \quad \Longrightarrow \quad \boxed{\partial_\nu T_p^{\mu\nu} = F^{\mu\alpha} J_\alpha}$$



- ➢ 设 V 是包含某个粒子 e 的极小区域

$$\begin{cases} \int_V dV\, \partial_\nu T_p^{\mu\nu} = \int_V dV\, \partial_0 T_p^{\mu 0} + \boxed{\int_V dV\, \partial_k T_p^{\mu k}} = \frac{dp^\mu}{dt} = \boxed{\frac{1}{\gamma}\frac{dp^\mu}{d\tau}} \\ \qquad \| \\ \int_V dV\, F^{\mu\alpha} J_\alpha = F^{\mu\alpha} \int_V dV\, J_\alpha = \frac{1}{\gamma} F^{\mu\alpha} u_\alpha \underbrace{\boxed{\int_V dV\, \rho}}_{e} = \boxed{\frac{1}{\gamma} e F^{\mu\alpha} u_\alpha} \end{cases}$$

由此，我们就证明了带电粒子的动力学方程为

$$\frac{dp^\mu}{d\tau} = e F^{\mu\alpha} u_\alpha$$

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